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LETTER TO THE EDITOR

A quantum version of the Nekhoroshev theorem*

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Abstract. Using a simple method, we prove that the quantum propagator for some polynomially perturbed harmonic oscillators is close to the propagator for the unperturbed oscillators for a small coupling constant and arbitrarily large times.

In this letter, we study the quantum propagator for polynomially perturbed harmonic oscillators for a small coupling constant ε. We are interested in the problem of when the difference between the propagator of the perturbed oscillators and the propagator of unperturbed oscillators is of order $O(\varepsilon^{\alpha})$ for large times. The Nekhoroshev theorem is the main result concerning the classical analogue of this problem (see [2]). The quantum case was studied by Herczyński [4], who obtained a number of results by using the Lie perturbation method. In this letter, using a simple method we establish a result which is in some sense stronger than one of the main results of [4].

The letter is organized as follows. First we introduce some notation which makes it possible to formulate the results in a form formally resembling the statement of the Nekhoroshev theorem. We then prove the main theorem. We close the letter with several remarks, including a comment on a semiclassical blow-up of the estimates obtained.

Let H_0 be a self-adjoint operator with purely discrete spectrum in a separable Hilbert space $\mathscr{H}$. Let $\{e_m\}_{m\in M}$ be an orthonormal basis consisting of eigenvectors of H_0 indexed by a countable set M. For each $m \in M$, let E_m be the eigenvalue corresponding to e_m. As usual, we let $\mathbb{R}_+ = \{r \in \mathbb{R}: r \geqslant 0\}$ and denote by $\mathbb{T}$ the group of complex numbers with unit modulus identified with the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. Let $U: \mathscr{H} \to \mathbb{R}_+^M \times \mathbb{T}^M$ be the map defined by

$$U(x) = (A(x), \varphi(x)) \qquad x \in \mathscr{H}$$

where $A(x) = (A_m(x))_{m\in M} \in \mathbb{R}_+^M$ and $\varphi(x) = (\varphi_m(x))_{m\in M} \in \mathbb{T}^M$ are uniquely determined by the identity

$$x = \sum_{m\in M} A_m(x) \exp(-\mathrm{i}\varphi_m(x)) e_m \qquad (1a)$$

or, more precisely, are given by

$$A_m(x) = |(x, e_m)| \qquad \varphi_m(x) = -\arg((x, e_n)) \qquad (1b)$$

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with the convention that $\arg(0)=0$. Let $\exp(-\mathrm{i}tH_0)$, $t\in\mathbb{R}$, be the propagator in $\mathscr{H}$ corresponding to H_0. Clearly, for each $x_0\in\mathscr{H}$ and each $t\in\mathbb{R}$,

$$x_t=\exp(-\mathrm{i}tH_0)x_0=\sum_{m\in M}A_m(x_0)\exp[-\mathrm{i}(\varphi_m(x_0)+tE_m)]e_m$$

whence

$$A(x_t)=A(x_0)\qquad \varphi(x_t)=\varphi(x_0)+tK(x_0) \tag{2}$$

where $K(x_0)\in\mathbb{R}^M$ is given by

$$(K(x_0))_m=\begin{cases}E_m & \text{if } A_m(x_0)\neq 0\\ 0 & \text{otherwise.}\end{cases}$$

We thus see that the evolution x_t is almost periodic in the 'coordinates' (A,φ). Following the terminology commonly used for integrable systems in classical mechanics, we shall call the mappings A and φ *the action coordinate* and *the angle coordinate* for the operator H_0, respectively.

Note that when all the E_m are simple, then the action coordinate is, up to a permutation of M, uniquely determined by the operator H_0. Moreover, the angle coordinate is uniquely determined up to a shift in the group $\mathbb{T}^M$ (for $x\in\mathscr{H}$ with $A_m(x)\neq 0$, $m\in M$), the ambiguity resulting from the freedom of choice of a normed eigenvector corresponding to a given eigenvalue of H_0 up to a scalar of unit modulus. Hereafter, when speaking about the angle variable of H_0, we shall always assume that an initial choice of the eigenvector basis has been made.

To compare values of the action coordinate for various $x\in\mathscr{H}$, we introduce the following metric ρ in Ran $A=\mathbb{R}_+^M\cap l^2(M)$:

$$\rho(C,D)=\left[\sum_{m\in M}(C_m-D_m)^2\right]^{1/2}. \tag{3}$$

If we let $\|\cdot\|$ be the norm in $\mathscr{H}$, then clearly, for any $x, y\in\mathscr{H}$,

$$\rho(A(x),A(y))\leqslant\|x-y\|. \tag{4}$$

Assume that H_0 is the Schrödinger operator in $L^2(\mathbb{R}^d)$ for non-resonant d-dimensional harmonic oscillators,

$$H_0=\frac{1}{2}\sum_{j=1}^{d}\left(-\hbar^2\frac{\partial^2}{\partial q_j^2}+\omega_j^2q_j^2-\hbar\omega_j\right) \tag{5}$$

with $(\omega_1,\ldots\omega_d)=\omega\in\mathbb{R}_+^d$ satisfying the condition

$$\omega\cdot\nu\neq 0 \text{ for } \nu\in\mathbb{Z}^d \qquad \nu\neq 0 \tag{6}$$

where $\omega\cdot\nu=\sum_{j=1}^d\omega_j\nu_j$. Let $|\nu|=\sum_{j=1}^d|\nu_j|$. The operator H_0 has purely discrete spectrum consisting of eigenvalues $E_0(\nu)=\hbar\omega\cdot\nu$, $\nu\in\mathbb{N}^d$. In view of (6), all the $E_0(\nu)$ are simple. As eigenvectors e_ν one can take suitable Hermite functions. Herczyński in [4] considers perturbations of H_0 of the form

$$H_\varepsilon=H_0+\varepsilon V \tag{7}$$

where $\varepsilon>0$ and V is the operator of multiplication by a real, bounded-below polynomial $V(q)$ of degree k. The sum in (7) is to be interpreted as the closure of the appropriate essentially self-adjoint operator on $C_0^\infty(\mathbb{R}^d)$. One of the main results of [4] is a kind of quantum version of the Nekhoroshev theorem.

Theorem 1. Let ω be such that

$$|\omega\cdot\nu|^{-1}\leqslant C_1|\nu|^{\gamma}\text{ for }\nu\in\mathbb{Z}^d\qquad\nu\neq0\tag{6'}$$

for some $C_1>0$ and $\gamma>0$. Let P_E be the orthogonal projection on span $\{e_\nu: E_0(\nu)\leqslant E\}$ and let K_j be the operator diagonal in the basis $\{e_\nu\}_{\nu\in\mathbb{N}^d}$, given by the equality $K_je_\nu=E_j(\nu)e_\nu$, where $E_j(\nu)$, $j=0,1,\ldots$, are the coefficients in the formal Rayleigh–Schrödinger series for the perturbed eigenvalue $E_\varepsilon(\nu)$ of the operator H_ε near $E_0(\nu)$. Then there exist $C>0$, $B>0$, and $D>0$ such that if $E\geqslant1$, $\hbar<\min\{1,E(k\max_{j=1,\ldots,d}\omega_j)^{-1}\}$ and $0<\varepsilon<B\varepsilon^*$, then there exists $r(\varepsilon)\in\mathbb{N}$ such that $K_\varepsilon=\Sigma_{j=0}^{r(\varepsilon)}\varepsilon^jK_j$ satisfies

$$\|(\exp(-\mathrm{i}tH_\varepsilon/\hbar)-\exp(-\mathrm{i}tK_\varepsilon/\hbar))P_E\|\leqslant C\hbar^{-\alpha}E^{k/2}\varepsilon^{\alpha}\tag{8}$$

for $|t|\leqslant\hbar\varepsilon^\alpha\exp[(\varepsilon_*/\varepsilon)^\alpha]$, where $\varepsilon_*=D\hbar E^{-k/2}$ and $\alpha=(\gamma+d+2+k/2)^{-1}$.

The proof of the above theorem relies on the adaptation of the Lie method to the quantum setting ([4]; see also [1]).

Theorem 1 can be reformulated in terms of action coordinate for the unperturbed operator H_0. With $M=\mathbb{N}^d$, let $A: L^2(\mathbb{R}^d)\to\mathbb{R}_+^M$ be the quantum action coordinate for H_0 such that $A_\nu(x)=|(x,e_\nu)|$, $\nu\in\mathbb{N}^d$. Denote, as before, $x_{\varepsilon,t}=\exp(-\mathrm{i}tH_\varepsilon/\hbar)x_0$ for $x_0\in L^2(\mathbb{R}^d)$ and $t\in\mathbb{R}$. The following is a simple consequence of theorem 1.

Theorem 2. Under the assumptions of theorem 1, if $x_0\in\operatorname{Ran}P_E$ and $\|x_0\|\leqslant1$, then

$$\rho(A(x_{\varepsilon,t})A(x_0))\leqslant C\hbar^{-\alpha}E^{k/2}\varepsilon^{\alpha}\tag{8'}$$

for $|t|\leqslant\hbar\varepsilon^\alpha\exp[(\varepsilon_*/\varepsilon)^\alpha]$.

Proof. Since K_ε is diagonal, we have $A(\exp(-\mathrm{i}tK_\varepsilon/\hbar)x_0)=A(x_0)$. Now, we obtain (8′) by invoking (4) and (8). □

Theorem 2 may be regarded as a still closer quantum analogue of the classical result of Nekhoroshev. However, as we shall prove, theorem 2 is far from being optimal. More precisely, under similar assumptions, the estimate of type (8′), i.e., $\rho(A(x_{\varepsilon,t}),A(x_0))\leqslant C(\hbar,E)\varepsilon^b$ is valid for arbitrarily large times. This result will be formulated as theorem 3.

Hereafter, retaining the assumptions from the theorem on asymptotic behaviour of Rayleigh–Schrödinger series (see [6, theorem XII.14]), we assume that

1^0. V is the operator of multiplication by a real, measurable function, such that $\{e_\nu\}_{\nu\in\mathbb{N}^d}\subset D(V)$ and the operator $H_\varepsilon=H_0+\varepsilon V$ is a certain self-adjoint extension of the above sum defined on span $\{e_\nu:\nu\in\mathbb{N}^d\}$†;

2^0. $\lim_{\varepsilon\to0+}\|(H_\varepsilon-z)^{-1}-(H_0-z)^{-1}\|=0$ for some $z\notin\sigma(H_0)$.

Theorem 3. If V satisfies 1^0 and 2^0, then for every $E>0$ there exist $\varepsilon_0>0$ and $\tilde{C}>0$ such that for $0<\varepsilon<\varepsilon_0$ and $x_0\in\operatorname{Ran}P_E$ with $\|x_0\|\leqslant1$

$$\rho(A(x_{\varepsilon,t}),A(x_0))\leqslant\tilde{C}\varepsilon\tag{8''}$$

for all $t\in\mathbb{R}$.

† This self-adjoint extension need not be unique; the existence of at least one such extension is guaranteed by a theorem of von Neumann (see [6, theorem X.3]).

Proof. Using arguments similar to those used in the proof of [6, theorem XII.14], we infer that for any $\nu \in \mathbb{N}^d$ there exists $\varepsilon(\nu)$ such that for $0 \leqslant \varepsilon \leqslant \varepsilon(\nu)$ the operators

$$P_\varepsilon(\nu) = \frac{-1}{2\pi\mathrm{i}} \int_{|z-E_0(\nu)|=\delta_\nu} (H_\varepsilon - z)^{-1}\,\mathrm{d}z$$

are well defined and converge in norm to $P_0(\nu)$ as $\varepsilon \to 0+$, provided δ_ν is chosen so that

$$\sigma(H_0) \cap [E_0(\nu) - 2\delta_\nu, E_0(\nu) + 2\delta_\nu] = \{E_0(\nu)\}. \tag{9}$$

Each $P_\varepsilon(\nu)$ is the projection onto the space spanned by the (usually non-normalized) eigenvector $P_\varepsilon(\nu)e_\nu$ of H_ε, denoted $e_{\nu,\varepsilon}$, which corresponds to the unique eigenvalue $E_\varepsilon(\nu)$ satisfying $\|E_0(\nu) - E_\varepsilon(\nu)\| < \delta_\nu$. By 1^0 and the second resolvent formula,

$$\begin{aligned}
\|e_{\nu,\varepsilon} - e_\nu\| &= \|(P_\varepsilon(\nu) - P_0(\nu))e_\nu\| \\
&\leqslant \frac{1}{2\pi} \left\| \int_{|z-E_0(\nu)|=\delta_\nu} ((H_0 - z)^{-1} - (H_\varepsilon - z)^{-1})e_\nu\,\mathrm{d}z \right\| \\
&\leqslant \varepsilon\delta_\nu \sup_{|z-E_0(\nu)|=\delta_\nu} \|(H_\varepsilon - z)^{-1} V (H_0 - z)^{-1} e_\nu\| \\
&\leqslant \varepsilon\delta_\nu \|Ve_\nu\| \sup_{|z-E_0(\nu)|=\delta_\nu} (|E_0(\nu) - z|^{-1} \|(H_\varepsilon - z)^{-1}\|).
\end{aligned}$$

Note that, by 2^0, $(H_\varepsilon - z)^{-1}$ converges in norm to $(H_0 - z)^{-1}$ as $\varepsilon \to 0+$, uniformly for z from an arbitrary compact subset of the resolvent set of H_0. Now, one can choose $\tilde{\varepsilon}(\nu)$ with $0 < \tilde{\varepsilon}(\nu) \leqslant \varepsilon(\nu)$ so that, for each $0 < \varepsilon < \tilde{\varepsilon}(\nu)$,

$$\|e_{\nu,\varepsilon} - e_\nu\| \leqslant \tfrac{3}{2}\varepsilon \|Ve_\nu\| \sup_{|z-E_0(\nu)|=\delta_\nu} \|(H_0 - z)^{-1}\|.$$

By (9),

$$\|e_{\nu,\varepsilon} - e_\nu\| \leqslant \tfrac{3}{2}\delta_\nu^{-1} \|Ve_\nu\| \varepsilon. \tag{10}$$

Given $E > 0$, let $\varepsilon_0 = \inf_{E_0(\nu) \leqslant E} \tilde{\varepsilon}(\nu)$ and let $x_0 \in \operatorname{Ran} P_E$ be such that $\|x_0\| \leqslant 1$. Letting $x_0 = \sum_{E_0(\nu) \leqslant E} c_\nu e_\nu$, for any $\varepsilon \in (0, \varepsilon_0)$ and any $t \in \mathbb{R}$, denote

$$\tilde{x}_{\varepsilon,0} = \sum_{E_0(\nu) \leqslant E} c_\nu e_{\nu,\varepsilon}$$

$$\tilde{x}_{\varepsilon,t} = \exp(-\mathrm{i}tH_\varepsilon/\hbar)\tilde{x}_{\varepsilon,0} = \sum_{E_0(\nu) \leqslant E} c_\nu \exp(-\mathrm{i}tE_\varepsilon(\nu)/\hbar) e_{\nu,\varepsilon}$$

and

$$\hat{x}_{\varepsilon,t} = \sum_{E_0(\nu) \leqslant E} c_\nu \exp(-\mathrm{i}tE_\varepsilon(\nu)/\hbar) e_\nu.$$

Since $\exp(-\mathrm{i}tH_\varepsilon/\hbar)$ is unitary, we have

$$\|x_{\varepsilon,t} - \tilde{x}_{\varepsilon,t}\| = \|x_0 - \tilde{x}_{\varepsilon,0}\|$$

and therefore

$$\|x_{\varepsilon,t} - \tilde{x}_{\varepsilon,t}\| \leqslant \sum_{E_0(\nu) \leqslant E} |c_\nu| \|e_\nu - e_{\nu,\varepsilon}\| \leqslant \left(\sum_{E_0(\nu) \leqslant E} \|e_\nu - e_{\nu,\varepsilon}\|^2 \right)^{1/2}. \tag{11}$$

Analogously we have

$$\begin{aligned}
\|\tilde{x}_{\varepsilon,t} - \hat{x}_{\varepsilon,t}\| &= \left\| \sum_{E_0(\nu) \leqslant E} \exp(-\mathrm{i}tE_\varepsilon(\nu)/\hbar) c_\nu (e_\nu - e_{\nu,\varepsilon}) \right\| \\
&\leqslant \left(\sum_{E_0(\nu) \leqslant E} \|e_\nu - e_{\nu,\varepsilon}\|^2 \right)^{1/2}.
\end{aligned} \tag{12}$$

Furthermore $A(\hat{x}_{\varepsilon,t}) = A(x_0)$, so using (4), (10), (11), and (12), we obtain (8″) with

$$\tilde{C} = 3\left[\sum_{E_0(\nu)\leqslant E} \delta_\nu^{-2} \| Ve_\nu \|^2\right]^{1/2}. \tag{13}$$

□

Remarks

1. The fact that the estimate (8′, 8″) is valid for arbitrarily large times may be surprising at first, as this is much better than the estimate in the classical Nekhoroshev theorem. One can heuristically explain this phenomenon by noting that the classical counterpart of the case described by theorem 3 is a very rare instance in classical mechanics when the perturbed system is integrable for all $\varepsilon \in (0, \tilde{\varepsilon})$. The quantum analogue of the classical integrability is the pure discreteness of the spectrum of the corresponding unperturbed Schrödinger operator. A technical explanation of the 'overperformance' in theorem 3 goes as follows. In the proof of theorem 3, the unitary map $e_{\nu,\varepsilon} \to e_\nu$, which is close to the identity in a suitable operator topology, is used to diagonalize H_ε. In contrast, in the proof of theorem 1, a unitary operator which is a small perturbation of the identity, determined by means of the Lie method, brings H_ε to the form $K_\varepsilon + R_\varepsilon$ with K_ε diagonal and R_ε small but different from zero. It is the presence of the latter summand that is responsible for the estimates (8) and (8′) in theorems 1 and 2 being valid for long, but not arbitrarily long, times.

2. Interesting examples of potentials V to which theorem 3 can be applied are provided by polynomials. For conditions 1^0 and 2^0 to be satified, it is sufficient that V be a polynomial positive on the unit sphere, homogeneous, and of even degree; this follows from [6, section XII.3] (see also [3] and [7]). Note that, despite suggestions in [1] and [4], it is not proved in [6], [3] and [7] that it is enough to assume V to be a bounded-below polynomial only. On the other hand, bounded potentials satisfy the assumptions of theorem 3.

3. It is rather difficult to estimate the dependence of ε_0, in theorem 3, on $\hbar$ and E. For the constant $\tilde{C}$ this is easier, if we assume (6′). We then have

$$\delta_\nu \geqslant \tilde{B} E^{-\gamma} \hbar^{1+\gamma}$$

hence, if V is a polynomial of degree k, for $\hbar < 1$ and $E > k \max_{i=1,\ldots d} \omega_i$, we obtain

$$\tilde{C} \leqslant \tilde{D} E^{\gamma + k/2 + d/2} \hbar^{-(1+\gamma+d/2)}$$

where $\tilde{D}$ and $\tilde{B}$ depend only on ω and V. This inequality is a simple consequence of (13) and the fact that $\| VP_E \| \leqslant \tilde{D} E^{k/2}$ (see [4]). So, from the point of view of $\hbar$, the estimate (8′) is better than (8″); however, from the point of view of ε, (8″) is better, because $\alpha < \frac{1}{3}$ in theorem 2.

4. It seems unnecessary to assume in theorems 1 and 2 that the suitable Rayleigh–Schrödinger series is Borel summable to the perturbed eigenvalue, or even that V is bounded-below (see [4]). Similarly it is not essential that $\varepsilon > 0$. So these theorems can be applied to a larger class of potentials than theorem 3. Obviously, the assertion of theorem 3 is valid also for $\varepsilon < 0$ (with $|\varepsilon|$ at the right-hand side of (8″)) if '$\lim_{\varepsilon \to 0+}$' in 1^0 is replaced by '$\lim_{\varepsilon \to 0}$' (such an assumption restricts severely the class of admissible potentials).

5. From the technical point of view, the choice of the metric ρ (see (3)) seems to be natural. However, from the point of view of the transition from classical to quantum

mechanics, another choice may be more natural, for instance

$$\tilde{\rho}(C, D) = \left[\sum_{\nu \in \mathbb{N}^d} (C_\nu - D_\nu)^2 (E_0(\nu) + \tfrac{1}{2}\hbar|\omega|) \right]^{1/2}.$$

6. Another typical example of a classical integrable system to which the Nekhoroshev theorem can be applied is the Hamiltonian of free rotators. It would be interesting to establish an analogue of theorem 3 in this case. The Schrödinger operator for non-resonant free rotators has also purely discrete spectrum but the eigenvalues are not simple (see, e.g., [5]). This fact is a source of difficulties. We have a freedom of choice of quantum action coordinates and it is easy to prove that a result similar to theorem 3 holds if the choice is made appropriately. However, such a choice essentially depends on the perturbation V, and so the corresponding result is rather distant in nature from the Nekhoroshev theorem.

7. The constants in theorems 2 and 3 depend on $\hbar$ in such a way that the semicalssical limit $\hbar \to 0$ does not exist. Accordingly, it is only with abuse of language that one may dub these results quantum Nekhoroshev theorems. It remains an open problem to prove a theorem that could deservedly be qualified as a quantum Nekhoroshev theorem.

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